

STABLE MEASURES AND CENTRAL LIMIT THEOREMS IN SPACES OF STABLE TYPE¹

BY

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ABSTRACT. Let X be a symmetric random variable with values in a quasi-normed linear space E . X satisfies the central limit theorem on E with index p , $0 < p < 2$, if $\mathcal{L}(n^{-1/p}(X_1 + \cdots + X_n))$ converges weakly to some probability measure on E . Hoffman-Jørgensen and Pisier have shown that Banach spaces of stable type 2 provide a natural environment for the central limit theorem with index $p = 2$. In this paper we show that, for $0 < p < 2$, quasi-normed linear spaces of stable type p provide a natural environment for the central limit theorem with index p . A similar result holds also for the weak law of large numbers with index p .

0. Introduction. Let X be a symmetric random variable with values in a quasi-normed linear space E . We say that X satisfies the *central limit theorem* on E with index p , $0 < p \leq 2$ (in short $X \in \text{CLT}_p$), if $\mathcal{L}(n^{-1/p}(X_1 + \cdots + X_n))$ converges weakly to some probability measure, say ν , on E . Here X_1, X_2, \dots are independent copies of X . It is known (see, e.g., [3, Theorem 2]) that such limiting measures ν must be stable. If we want to specify the limit measure ν for an $X \in \text{CLT}_p$ we say that X is in the *domain of normal attraction* of ν .

Hoffman-Jørgensen and Pisier [9] have shown that there is a special relationship between CLT_2 and spaces of stable type 2. That is, the conditions $EX = 0$, $E\|X\|^2 < \infty$ (that pertain even when E is the real line) imply $X \in \text{CLT}$ if and only if E is of stable type 2. This led us to pose the following question: Do spaces of stable type p have a special role in relation to CLT_p for $0 < p < 2$, analogous to the role spaces of stable type 2 play for CLT_2 ? The answer to this question is yes. In the case of a Banach space E it is contained in the following theorem:

THEOREM 0.1. *Let $0 < p < 2$. The following properties of a Banach space E are equivalent:*

- (i) *E is of stable type p .*

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(ii) For all random vectors X with values in E $\lim_{a \rightarrow \infty} a^p P(\|X\| > a) = 0$ iff $\mathcal{L}(n^{-1/p}(X_1 + \cdots + X_n)) \rightarrow \delta_0$ weakly as $n \rightarrow \infty$, where X_1, X_2, \dots are independent copies of X and δ_0 is the probability measure concentrated at the origin.

(iii) Let η_1, η_2, \dots be independent, symmetric, real valued random variables, θ a real valued stable random variable with characteristic function $\exp(-|t|^p)$, $\theta_1, \theta_2, \dots$ independent copies of θ and $c_p = [\int_0^\infty ((\sin x)/x^p) dx]^{-1}$. Then given any sequence $X_1, X_2, \dots \in E$ such that $Y = \sum \eta_i x_i$ converges a.s. we have that

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i \in A} \eta_i x_i \right\| > a \right] = c_p \sum_{i \in A} \|x_i\|^p < \infty,$$

for all subsets A of the positive integers if and only if $Z = \sum \theta_i x_i$ converges a.s. and Y satisfies $\mathcal{L}(n^{-1/p}(Y_1 + \cdots + Y_n)) \rightarrow \mathcal{L}(Z)$ weakly as $n \rightarrow \infty$, where Y_1, Y_2, \dots are independent copies of Y .

This theorem was announced in [20]. It will be proved in a more general setting as Theorem 5.1. In considering spaces classified into spaces of stable type p for $0 < p < 1$ it is natural and essential to consider general quasi-normed spaces such as l^q , $0 < q < 1$. Otherwise the classification is meaningless since all Banach spaces are of stable type p for $p < 1$. Therefore we do not restrict our attention only to Banach spaces but work in the more general context of quasi-normed linear spaces.

§1 is devoted to precise definitions of the terms we use and to certain preliminary facts. §2 deals with the representation of stable random vectors as random integrals and the relationship between these random integrals and their characteristic functionals. These results indicate why spaces of stable type p are a natural environment for stable measures of exponent p . Statement (ii) of Theorem 0.1 is evidently a weak law of large numbers. They will be studied in §3. §4 is technical. It contains extensions to quasi-normed linear spaces of integrability and comparison theorems that were obtained in [8], [10] on Banach spaces. It also treats the integrability of concave functions of random vectors. In §5 we prove Theorem 5.1, the extension of Theorem 0.1 to quasi-normed linear spaces. §6 contains more results on stable measures and their domains of normal attraction and deals with some concrete examples.

The value of our results is that they are quite general. Results extending the classical theorems on the real line have been found in finite dimensional spaces by Levy and Gnedenko [6], Rvaceva [28] and on Hilbert spaces by Kuelbs and Mandrekar [17] and Klosowska [13]. In §6 we use Theorem 3.1 [16] of Kuelbs and Mandrekar. This result played a much more central role in the early stages of our work and helped us to clarify our problem. We gratefully acknowledge the help of the referee.

1. Preliminaries. Let E be a real, separable, quasi-normed complete linear space equipped with a quasi-norm $\| \cdot \|$, i.e., a function from E to the positive real numbers such that $\|x\| = 0$ iff $x = 0$; $\|\alpha x\| = |\alpha| \|x\|$, $\alpha \in \mathbb{R}$, $x \in E$; $\|x + y\| \leq A(\|x\| + \|y\|)$ where $x, y \in E$ and A is a positive real number. Such a space is pseudo-convex and consequently it also admits an r -homogeneous norm $\| \cdot \|_r$, $0 < r \leq 1$, which is equivalent (topologically) to the quasi-norm $\| \cdot \|$ (see [35]). Therefore, without any loss of generality, we shall assume that there exists an r , $0 < r \leq 1$, such that $\| \cdot \|_r \equiv \| \cdot \|^r$ satisfies the triangle inequality, i.e., $\|x + y\|_r \leq \|x\|_r + \|y\|_r$, $x, y \in E$. E^* will denote the topological dual of E and we shall always assume that E^* separates the points of E .

Throughout this paper θ will denote a real valued stable random variable with characteristic function $E \exp i t \theta = \exp - |t|^p$ and $\{\theta_i\}$ independent copies of θ .

DEFINITION 1.1. E is said to be of stable type p , $0 < p \leq 2$, if for each $q < p$ there exists a constant C such that for all integers n and any $x_1, \dots, x_n \in E$

$$\left(E \left\| \sum_{i=1}^n \theta_i x_i \right\|^q \right)^{1/q} \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}. \quad (1.1)$$

Actually by [7], [18] E is of stable type p if there exists a single $q \in [0, p)$ such that the above inequality holds. In the case $q = 0$ the left-hand side of (1.1) should be replaced by any of the L^0 -gauges. On the other extreme one can replace the L^q norms in Definition 1.1 by the Lorentz norm Λ_p . (For a random vector X with values in E , $\Lambda_p(X) = \sup_{t>0} t^{1/p} (\|X\| > t)$.)

REMARK 1.1. It is a simple consequence of the closed graph theorem in quasi-normed spaces (Theorem 2.7.I [33] and Theorem 6.1 [7]) that E is of stable type p if and only if for all sequences x_1, x_2, \dots of elements of E for which $\sum \|x_i\|^p < \infty$ the series $\sum \theta_i x_i$ converges a.s. (or equivalently converges in L^q , $q < p$).

If E is of stable type p , and if ξ_1, \dots, ξ_n are independent with characteristic functions $\exp - \alpha_i |t|^p$, $i = 1, \dots, n$, then for any $x_1, \dots, x_n \in E$

$$\left(E \left\| \sum_{i=1}^n \xi_i x_i \right\|^q \right)^{1/q} \leq C \left(\sum_{i=1}^n \alpha_i \|x_i\|^p \right)^{1/p} \quad (1.2)$$

where $q < p$ and $C = C(q)$.

If $q \geq 2$ then l_q, L_q are examples of spaces of stable type 2. If $0 < p < q$ and $0 < p < 2$ then l_q and L_q are of stable type p but l_p and L_p are not of stable type p . If $p < 1$ then all Banach spaces are of stable type p .

A theorem of Maurey and Pisier [22] and Krivine [15] states that a Banach

space E is of stable type p , $p < 2$, if and only if l_p is not finitely representable in E . This gives a particularly nice, purely geometric characterization of spaces of stable type p . Recall that l_p is finitely representable in E if for each $\varepsilon > 0$ and each $n = 1, 2, \dots$ one can find $x_1, \dots, x_n \in E$ such that for all $\alpha_1, \dots, \alpha_n \in \mathbf{R}$,

$$\left(\sum_{i=1}^n |\alpha_i|^p \right)^{1/p} \leq \left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq (1 + \varepsilon) \left(\sum_{i=1}^n |\alpha_i|^p \right)^{1/p}.$$

Spaces of stable type 1 are precisely those which are Beck-convex [1], [26]. A Banach space E is Beck-convex if there exists an $\varepsilon > 0$ and an integer n such that for any $x_1, \dots, x_n \in S_E$, the unit sphere of E , $\|x_1 \pm x_2 \pm \dots \pm x_n\| \leq (1 - \varepsilon)n$ for some choice of \pm .

On any quasi-normed space E for each p , $0 < p < 2$ and $0 < q < p$, there exists a constant C_1 such that for all integers n and $x_1, \dots, x_n \in E$

$$C_1 \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq \left[E \left\| \sum_{i=1}^n \theta_i x_i \right\|^q \right]^{1/q} \quad (1.3)$$

(see [21]).

We shall also refer to spaces of Rademacher type p , $0 < p < 2$. E is said to be of Rademacher type p , $0 < p < 2$, if there exists a constant C_2 such that for all integers n and any $x_1, \dots, x_n \in E$

$$\left[E \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \right]^{1/p} \leq C_2 \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \quad (1.4)$$

where $\varepsilon_1, \dots, \varepsilon_n$ are Rademacher functions, that is independent identically distributed random variables with $P(\varepsilon_1 = \pm 1) = \frac{1}{2}$. It follows from (1.4) that if X_1, X_2, \dots are symmetric random vectors with values in a space of Rademacher type p then

$$\left[E \left\| \sum_{i=1}^n X_i \right\|^p \right]^{1/p} \leq C_2 \left(\sum_{i=1}^n E \|X_i\|^p \right)^{1/p}. \quad (1.5)$$

For more information on types of normed spaces see [31].

Let the triple (S, Σ, m) denote a measurable space with measure m and $L^p(S, \Sigma, m; E)$ the completion of the (S, Σ) simple functions with values in E under the quasi-norm

$$\|f\|_{L^p} = \left(\int \|f(s)\|^p m(ds) \right)^{1/p}. \quad (1.6)$$

We will sometimes abbreviate $L^p(S, \Sigma, m; E)$ by $L^p(E)$ or $L^p(m; E)$.

A random vector X (i.e., a strongly measurable mapping) on a probability space (Ω, \mathcal{F}, P) with values in E , or equivalently its distribution $\mu = \mathcal{L}(X)$, is called symmetric stable or just stable if for any $\alpha, \beta > 0$ and independent copies X_1, X_2 of X there exists a $\gamma > 0$ such that

$$\mathcal{L}(\alpha X_1 + \beta X_2) = \mathcal{L}(\gamma X). \quad (1.7)$$

In (1.7) we always have $\gamma = (\alpha^p + \beta^p)^{1/p}$ for some $p \in (0, 2]$. In this case X and μ are called stable of exponent p .

Since we assume that E^* separates points in E an E -valued random vector is stable if and only if, for each $x^* \in E^*$, x^*X is a real stable random variable [5]. In this case the characteristic functional of any stable random vector X has the representation

$$E \exp ix^*X = \exp\left(-\int_E |x^*x|^p \sigma(dx)\right) \quad (1.8)$$

[29], where $0 < p \leq 2$ is the exponent of X and σ is a finite Borel measure on E such that $\int_E \|x\|^p \sigma(dx) < \infty$. The measure σ can always be chosen to be concentrated on the unit sphere thereby satisfying the integrability condition. We shall refer to σ as the spectral measure of X , or equivalently, μ ,

Discrete spectral measures $\sum \alpha_i \delta_{\{x_i\}}$ correspond to stable random vectors of the form $\sum |\alpha_i|^{1/p} \theta_i x_i$. In this case, to avoid reduction to lower dimensional cases we assume that

$$x_i \notin \overline{\text{span}} [x_j; j = 1, 2, \dots; i \neq j], \quad (1.9)$$

$i = 1, 2, \dots$

A set K is absolutely r -convex if, for any $\alpha_1, \dots, \alpha_n \in R$; $x_1, \dots, x_n \in K$, $\sum |\alpha_i|^r = 1$ implies $\sum \alpha_i x_i \in K$.

2. Random integrals and weak convergence of stable measures on spaces of stable type p . We begin with the construction of random integrals of the form $\int f dM$ where f is deterministic and E -valued and M is a stable real random measure. For real valued f and even more general M this construction can be found in [2] and [30].

Let m be a nonnegative, finite measure on a measurable space (S, Σ) and let M be a stable independently scattered random measure of exponent p on Σ for which m is the control measure, that is, for each $A \in \Sigma$, $E \exp itM(A) = \exp(-m(A)|t|^p)$. Recall that a mapping $M: \Sigma \rightarrow L^0(\Omega, \mathcal{F}, P)$ is called an independently scattered random measure if for any pairwise disjoint $A_1, A_2, \dots \in \Sigma$ the random variables $M(A_1), M(A_2), \dots$ are stochastically independent and $M(\cup A_i) = \sum M(A_i)$, where the series on the right converges in probability. It follows from Kolmogorov's consistency theorem that given such a control measure m one can construct the related independently scattered random stable measure M .

If $f: S \rightarrow E$ is a simple function, i.e., $f = \sum x_i I_{A_i}$, where $A_i \in \Sigma$ are pairwise disjoint and $x_i \in E$ (I_A denotes the indicator function of the set A) we set

$$\int_S f(s) M(ds) = \sum x_i M(A_i). \quad (2.1)$$

In this case $\int f dM$ is a stable E -valued random vector. For f simple and in $L^p(S, \Sigma, m; E)$ and for $q < p$ the map $f \rightarrow \int_S f(s) M(ds)$ is a linear operator with values in $L^q(\Omega, \mathcal{F}, P; E)$. By (1.2) we see that

$$\begin{aligned} \left\| \int f dM \right\|_{L^q} &= \left(E \left\| \sum x_i M(A_i) \right\|^q \right)^{1/q} \\ &\leq C \left(\sum m(A_i) \|x_i\|^p \right)^{1/p} = C \left(\int \|f(s)\|^p m(ds) \right)^{1/p}. \end{aligned} \quad (2.2)$$

Since the simple functions are dense in $L^p(S, \Sigma, m; E)$ there exists a unique extension of this operator onto the whole of L^p . This extension will be denoted also by $\int f dM$, $f \in L^p$ and satisfies (2.2). Summarizing we get

THEOREM 2.1. *If E is of stable type p and M is a stable independently scattered random measure of exponent p on (S, Σ) with control measure m , then, for each q , $0 < q < p \leq 2$, there exists a linear map*

$$L^p(S, \Sigma, m; E) \ni f \rightarrow \int_S f dM \in L^q(\Omega, \mathcal{F}, P; E) \quad (2.3)$$

the values of which are stable random vectors in E satisfying

$$\left(E \left\| \int f dM \right\|^q \right)^{1/q} \leq C \left(\int \|f\|^p dm \right)^{1/p} \quad (2.4)$$

for some finite constant C independent of f .

REMARK 2.1. The inequality (2.4) is also valid if the L^q norm is replaced by the Lorentz norm $\Lambda_p(\int f dM)$.

It is quite easy to compute the characteristic functional of $\int f dM$. Indeed, take simple functions $f_n = \sum_i x_i^n I_{A_i^n}$ converging to f in $L^p(m; E)$. Then because of (2.4), $\mathcal{L}(\int f_n dM) \rightarrow \mathcal{L}(\int f dM)$ weakly. Therefore for each $x^* \in E^*$

$$\begin{aligned} E \exp ix^* \int f dM &= \lim_n E \exp \left[ix^* \sum x_i^n M(A_i^n) \right] \\ &= \lim_n \prod_i \exp(-m(A_i^n) |x^* x_i^n|^p) \\ &= \lim_n \exp \left(- \int |x^* f_n(s)|^p m(ds) \right) \\ &= \exp \left(- \int_S |x^* f(s)|^p m(ds) \right). \end{aligned}$$

We use this in the following corollary which, in the particular case $S = E$ and $f(x) = x$, is due to Mouchtari [23].

COROLLARY 2.2. *The following two properties are equivalent:*

(2.5) *E is of stable type p , $0 < p \leq 2$:*

(2.6) *For any finite measure space (S, Σ, m) and any $f \in L^p(S, \Sigma, m; E)$ the function*

$$E^* \ni x^* \rightarrow \exp\left(-\int_S |x^* f(s)|^p m(ds)\right) \in \mathbb{R}$$

is the characteristic functional of a stable measure $\mu = \mathcal{L}(f dM)$ on E .

PROOF. Theorem 2.1 and the computation of the characteristic functional of $\mathcal{L}(f dM)$ given above show that (2.5) implies (2.6).

Now, assume (2.6) and let x_1, x_2, \dots be a sequence of elements of E such that $\sum \|x_i\|^p < \infty$. Take $S = E$ with Σ being the Borel sets of E . Let m be concentrated on x_1, x_2, \dots and such that $m(\{x_i\}) = \|x_i\|^p$. Finally take $f(x) = x/\|x\|$. By (2.6), $\varphi(x^*) = \exp[-\sum |x^* x_i|^p]$ is the characteristic functional of $\mathcal{L}(\sum \theta_i x_i)$. Since the summands $\theta_i x_i$ are symmetric and independent and because for each $x^* \in E^*$

$$\lim_n \exp\left[-\sum_{i=1}^n |x^* x_i|^p\right] = \varphi(x^*),$$

by Ito-Nisio's theorem [31, II, p. 274], which extends easily to spaces with r -homogeneous norms and separating duals, we obtain the a.s. convergence of the series $\sum \theta_i x_i$. That means E is of stable type p .

REMARK 2.2. In representing all stable measures μ on spaces E of stable type p we can restrict our attention to measures of the form $\mu = \mathcal{L}(\int_{S_E} x M(dx))$ where the control measure m of M is concentrated on the unit sphere S_E of E . This follows by a simple change of variables.

Spaces of stable type p have the property that the weak convergence of spectral measures implies the weak convergence of the corresponding stable measures. The following theorem makes this precise. We will not use this theorem in this paper but we would like to state it here for future reference. A somewhat weaker version was proved for $p = 2$ by Rosinski and Woyczynski [32].

THEOREM 2.3. *The following conditions are equivalent:*

(2.7) *E is of stable type p , $0 < p \leq 2$.*

(2.8) *If $\sigma_1, \sigma_2, \dots$ are spectral measures such that σ_i converges weakly to σ_∞ on S_E and if μ_1, μ_2, \dots are the corresponding stable measures on E then there exists a stable measure μ_∞ and μ_i converges weakly to μ_∞ on E .*

PROOF. (2.7) implies (2.8). Let $M_n, n = 1, 2, \dots, \infty$, be stable independently scattered random measures of exponent p on S_E with control measures σ_n . Let $\varepsilon > 0$. By Prokhorov's theorem we can find a compact set $K \subset S_E$ such that, for all $n = 1, 2, \dots, \infty$; $\sigma_n(S_E/K) < \varepsilon$. Let $f = \sum x_i I_{A_i}$ be simple functions with finite range such that the A_i are continuity sets of σ_∞ , $\|f(x) - x\| < \varepsilon$ on K and $\|f(x)\| < 1$ elsewhere. Then

$$\begin{aligned} \int_{S_E} \|x - f(x)\|^p \sigma_n(dx) &= \int_K \|x - f(x)\|^p \sigma_n(dx) \\ &\quad + \int_{S_E/K} \|x - f(x)\|^p \sigma_n(dx) \\ &< \varepsilon^p + (2A)^p \varepsilon \end{aligned}$$

for all $n = 1, 2, \dots, \infty$ where A is the constant associated with the quasi-norm. By Theorem 2.1, we have that if $q < p$

$$\begin{aligned} \left[E \left\| \int_{S_E} (x - f(x)) M_n(dx) \right\|^q \right]^{1/q} &< C \left(\int_{S_E} \|x - f(x)\|^p \sigma_n(dx) \right)^{1/p} \\ &< (\varepsilon^p + (2A)^p \varepsilon)^{1/p} \end{aligned}$$

for all $n = 1, 2, \dots, \infty$.

Therefore, because $\mu_n = \mathcal{L}(x M_n(dx))$, and

$$\begin{aligned} \int x M_n(dx) - \int x M_\infty(dx) &= \int (x - f(x)) M_n(dx) + \int (f(x) - x) M_\infty(dx) \\ &\quad + \int f(x) M_n(dx) - \int f(x) M_\infty(dx), \end{aligned}$$

it is sufficient to show that $\mathcal{L}(\int_{S_E} f(x) M_n(dx)) \rightarrow \mathcal{L}(\int f(x) M_\infty(dx))$ weakly as $n \rightarrow \infty$. However, $\int f(x) M_n(dx)$ are stable random vectors taking values in a fixed finite dimensional subspace of E spanned by the values of f . Therefore to prove weak convergence it is sufficient to prove convergence of the characteristic functionals. For each $x^* \in E^*$

$$\begin{aligned} E \exp \left[i x^* \int f(x) M_n(dx) \right] &= \exp \left[- \int_{S_E} |x^* f(x)|^p \sigma_n(dx) \right] \\ &= \exp \left[- \sum |x^* x_j|^p \sigma_n(A_j) \right]_{n \rightarrow \infty} \exp \left[- \sum |x^* x_j|^p \sigma_\infty(A_j) \right] \\ &= E \exp \left[i x^* \int f(x) M_\infty(dx) \right] \end{aligned}$$

because A_j 's are continuity sets for σ_∞ and $\sigma_n \rightarrow \sigma_\infty$ weakly. Clearly the measure $\mu_\infty = \mathcal{L}(\int x M_\infty(dx))$.

If E is not of stable type p we can find $x_1, x_2, \dots, \sum_{i=1}^\infty \|x_i\|^p < \infty$ but $\sum_{i=1}^\infty \theta_i x_i$ does not converge. We take for σ_∞ the measure concentrated on

$x_i/\|x_i\|$ such that $\sigma_\infty(\{x_i/\|x_i\|\}) = \|x_i\|^p, i = 1, 2, \dots$, and for σ_n the measure concentrated on $x_i/\|x_i\|, i = 1, \dots, n$ such that $\sigma_n(\{x_i/\|x_i\|\}) = \|x_i\|^p, i = 1, \dots, n$. Clearly $\sigma_n \rightarrow \sigma_\infty$ weakly on S_E as $n \rightarrow \infty$. Consider the corresponding stable measures $\mu_n, n = 1, 2, \dots$. We have $\mu_n = \mathcal{L}(\sum_{i=1}^n \theta_i x_i)$. To say that the μ_n converges weakly is to say that $\sum_{i=1}^n \theta_i x_i$ converges weakly. However, by the extended Ito and Nisio theorem this would imply that $\sum_{i=1}^\infty \theta_i x_i$ converges a.s. and this is not true.

3. Weak law of large numbers. We prove an infinite dimensional analogue of the classical weak law of large numbers of Kolmogorov (see [4]). This result will be important in describing the domain of normal attraction of stable measures. The assumption that E is of stable type p , that appears below, is not only sufficient but also necessary as we will see in Theorem 5.1.

THEOREM 3.1. *Let E be of Rademacher type $p + \varepsilon, 0 < p < 2$, for some $\varepsilon > 0$ and X_1, X_2, \dots independent identically distributed random vectors with values in E . Then*

$$\frac{X_1 + \dots + X_n}{n^{1/p}} \rightarrow 0 \quad (3.1)$$

in probability if and only if

$$nP(\|X_1\| > n^{1/p}) \rightarrow 0. \quad (3.2)$$

Note. If E is a Banach space then E is of Rademacher type $p + \varepsilon, 0 < p < 2$, for some $\varepsilon > 0$ if and only if E is of stable type p .

PROOF. Define $S_n = X_1 + \dots + X_n$. Condition (3.1) implies (3.2) for X_1 with values in any quasi-normed linear space E . To see this note that because of symmetry $\|n^{-1/p}(\sum_{i \neq j} X_i + X_j)\|_r$ and $\|n^{-1/p}(\sum_{i \neq j} X_i - X_j)\|_r$ are equally distributed. It follows that

$$P\left(\left\|\frac{S_n}{n^{1/p}}\right\| > 2^{1-1/r\varepsilon}\right) > \frac{1}{2} P\left(\sup_{1 \leq j \leq n} \left\|\frac{X_j}{n^{1/p}}\right\| > \varepsilon\right). \quad (3.3)$$

Thus (3.1) implies $P(\sup_{1 \leq j \leq n} \|X_j\| > \varepsilon n^{1/p}) \rightarrow 0$ and this is equivalent to (3.2).

We now show that (3.2) implies (3.1). Let $Y_k = X_k I_{[\|X_k\| \leq n^{1/p}]}$, $Z_n = Y_1 + \dots + Y_n$ and $B_n = [S_n = Z_n]$. By assumption (and Theorem 1 [22], [25] in the case of Banach space E) there exists a $q, p < q < 2$, such that E is of Rademacher type q . Now, for each $\varepsilon > 0$,

$$\begin{aligned}
P(\|S_n/n^{1/p}\| > \varepsilon) & < P(B_n)P(\|S_n/n^{1/p}\| > \varepsilon|B_n) + P(B_n^c)P(\|S_n/n^{1/p}\| > \varepsilon|B_n^c) \\
& < P(\|Z_n/n^{1/p}\| > \varepsilon) + P(B_n^c) \\
& < \varepsilon^{-q}E\|Z_n/n^{1/p}\|^q + \sum_{k=1}^n P(\|X_k\| > n^{1/p}) \\
& < \varepsilon^{-q}n^{-q/p} \sum_{k=1}^n E\|X_k\|^q I_{[\|X_k\| < n^{1/p}]} + \sum_{k=1}^n P(\|X_k\| > n^{1/p}) \quad (3.4)
\end{aligned}$$

where at the last step we use (1.5). Continuing, (3.4)

$$< \varepsilon^{-q}n^{1-q/p} \int_{\|X_1\| < n^{1/p}} \|X_1\|^q dP + nP(\|X_1\| > n^{1/p}).$$

Thus we need to show that the first term, call it I_n , goes to zero as $n \rightarrow \infty$. Indeed

$$\begin{aligned}
I_n &= \varepsilon^{-q}n^{1-q/p} \sum_{k=1}^n \int_{k-1 < \|X_1\|^p < k} \|X_1\|^q dP \\
&< \varepsilon^{-q}n^{1-q/p} \sum_{k=1}^n k^{q/p} P(k-1 < \|X_1\|^p < k) \\
&< \varepsilon^{-q}n^{1-q/p} \sum_{k=1}^n \left(\sum_{i=1}^k i^{q/p-1} \right) P(k-1 < \|X_1\|^p < k) \\
&< \varepsilon^{-q}n^{1-q/p} \sum_{i=1}^n i^{q/p-1} P(i-1 < \|X_1\|^p < n) \\
&< \varepsilon^{-q}n^{1-q/p} \sum_{i=1}^n i^{q/p-1} P(\|X_1\|^p > i-1).
\end{aligned}$$

By (3.2) for each $\delta > 0$ we can find an integer i_0 , such that for all $i > i_0$, $iP(\|X_1\|^p > i-1) < \delta$. Therefore

$$\begin{aligned}
I_n &< \varepsilon^{-q}n^{1-q/p} \sum_{i=1}^{i_0} i^{q/p-1} P(\|X_1\|^p > i-1) \\
&\quad + \varepsilon^{-q}\delta n^{1-q/p} \sum_{i=i_0+1}^n i^{q/p-2}. \quad (3.5)
\end{aligned}$$

The last term in (3.5) is less than $(\delta/\varepsilon^q)C(p, q)$ where C is a constant that depends only on p and q . Since δ is arbitrary we get $P(\|S_n/n^{1/p}\| > \varepsilon) \rightarrow 0$ and this is true for all $\varepsilon > 0$. This completes the proof of Theorem 3.1.

As a corollary of the proof of Theorem 3.1 we obtain the following extension of a result which is best possible on the real line (see [24]).

THEOREM 3.2. *Let E be of Rademacher type p , $0 < p \leq 2$, and X_1, X_2, \dots independent, symmetric, random vectors with values in E . Then*

$$\frac{1}{n} \sum_{i=1}^n \int_{\|X_i\| < n} \|X_i\|^p dP \rightarrow 0 \quad (3.6)$$

and

$$\sum_{i=1}^n P(\|X_i\| > n^{1/p}) \rightarrow 0 \quad (3.7)$$

as $n \rightarrow \infty$ implies that

$$(X_1 + \dots + X_n)/n^{1/p} \rightarrow 0 \quad (3.8)$$

in probability as $n \rightarrow \infty$. Moreover if $p = 2$ then (3.8) implies (3.6) and (3.7).

A Banach space is of stable type 1 if and only if it is Beck-convex. Anticipating Theorem 5.1 we state the following corollary which shows that Beck-convex spaces are exactly those spaces in which an analogue of the classical weak law of large numbers (i.e., $p = 1$) depends on precisely those conditions necessary on the real line. This is a nice complement to Beck's "strong law of large numbers".

COROLLARY 3.3. *Let E be a Banach space; then the following are equivalent:*
(3.9) *E is Beck-convex.*

(3.10) *For all sequences X_1, X_2, \dots of independent, identically distributed, symmetric random vectors with values in E , $(X_1 + \dots + X_n)/n \rightarrow 0$ in probability if and only if $nP(\|X_1\| > n) \rightarrow 0$.*

4. Integrability and comparison theorems in quasi-normed spaces. In [8] and [10], Hoffman-Jørgensen and Jain and Marcus obtain results relating the expected value of functions of the norm of the sum of a series of independent Banach space valued random variables to the expected value of the norm of the supremum of the terms of the series. Also in [10] the expected value of convex functions of the norm of the sum of one such series is compared to the expected value of a related series. In §5 we will need generalizations of these results in two directions. First we replace the Banach space by a quasi-normed space E and secondly we consider comparison theorems for the expected value of certain concave functions of the quasi-norm.

We do not seek maximum generality but only those results which we use in this paper. Further generalizations can be obtained by our methods.

Let X_1, X_2, \dots be independent symmetric random variables with values in E . Assume that $S = \sum_{n=1}^{\infty} X_n$ exists and define $N = \sup_n \|X_n\|_r$, $F(u) = P(\|S\|_r > u)$, $G(u) = P(N > u)$. (Recall that $\|\cdot\|_r =^{\text{df}} \|\cdot\|^r$ is an r -homogeneous norm.)

The next two results are extensions of Lemma 3.4 and Theorem 3.3 in [10]. (Lemma 3.4 [8] is due to Hoffman-Jørgensen.)

LEMMA 4.1. For $\{X_n\}$, N , F and G as defined above

$$F(2t + s) \leq G(s) + 4F^2(t), \quad t, s > 0.$$

THEOREM 4.2. Let φ be a real valued increasing function satisfying $\varphi(0) = 0$ and $\varphi(2u) \leq c\varphi(u)$ for some $c > 0$. Assume that $\|S\| < \infty$ a.s. Then $E\varphi(N) < \infty$ if and only if $E\varphi(\|S\|_r) < \infty$. Furthermore if $E\varphi(\|S\|_r) < \infty$ then $\lim_{k \rightarrow \infty} E\varphi(\|\sum_{n=k}^{\infty} X_n\|_r) = 0$.

The proofs of these lemmas are exactly the proofs given in [10] since the only property of the Banach space that is used in those proofs is that it has a norm and the only property of the norm that is used is that it satisfies the triangle inequality. (Actually all of Theorem 3.3 [10] carries over.)

The extension of a version of the comparison theorems is not so transparent. We begin with a lemma which is analogous to Lemma 5.7 of [10].

LEMMA 4.3. Let ξ_1, ξ_2, \dots be independent, symmetric real valued random variables and $x_1, x_2, \dots \in E$. Let $\lambda_1, \lambda_2, \dots$ be real positive constants such that $\sup_k |\lambda_k| < 1$. Then

$$E \left(\sup_n \left\| \sum_{k=1}^n \lambda_k \xi_k x_k \right\|_r \right)^q \leq C_{r,q} E \left(\sup_n \left\| \sum_{k=1}^n \xi_k x_k \right\|_r \right)^q \quad (4.1)$$

for $0 < q \leq 1$, where $C_{r,q}$ is a constant depending only on r, q . If $\sum_{k=1}^{\infty} \xi_k x_k$ converges a.s. in E then also $\sum_{k=1}^{\infty} \lambda_k \xi_k x_k$ converges a.s. and we can replace $\sup_n \|\sum_{k=1}^n \cdot\|_r$ by $\|\sum_{k=1}^{\infty} \cdot\|_r$ in (4.1).

PROOF. The proof parallels the proof of Lemma 5.7 [10] except that instead of using the convexity of φ , as in Lemma 5.7 [10], we use the fact that for $0 < q \leq 1$, $a_i > 0$, $i = 1, \dots$, $(\sum a_i)^q \leq \sum a_i^q$.

A sequence ξ_1, ξ_2, \dots of symmetric real valued random variables is said to be uniformly nondegenerate if there exist constants $a, b > 0$ such that

$$P(|\xi_k| > a) \geq b, \quad k = 1, 2, \dots \quad (4.2)$$

Also as above let $\epsilon_1, \epsilon_2, \dots$ denote a Rademacher sequence.

The version of the comparison theorems that we use is:

THEOREM 4.4. Let $\eta_1, \eta_2, \dots; \xi_1, \xi_2, \dots$ be two sequences of independent symmetric real valued random variables such that for some $u_0 > 0$

$$P(|\eta_k| \geq u) \leq P(|\xi_k| \geq u) \quad (4.3)$$

for all $u \geq u_0$ and all $k = 1, 2, \dots$. Also assume that ξ_1, ξ_2, \dots is uniformly nondegenerate. Let $x_1, x_2, \dots \in E$ and assume that $\sum_{k=1}^{\infty} \xi_k x_k$ converges a.s. and $E \|\sum_{k=1}^{\infty} \xi_k x_k\|^q < \infty$ for some $q \leq r$. Then $\sum_{k=1}^{\infty} \eta_k x_k$ converges a.s. and

$$E \left\| \sum_{k=1}^{\infty} \eta_k x_k \right\|^q < C_1 |u_0|^q E \left\| \sum_{k=1}^{\infty} \varepsilon_k x_k \right\|^q + C_2 E \left\| \sum_{k=1}^{\infty} \xi_k x_k \right\|^q \quad (4.4)$$

where C_1, C_2 are constants depending only on r and q . Also under these hypotheses $\sum_{k=1}^{\infty} \varepsilon_k x_k$ converges a.s. and $E \|\sum_{k=1}^{\infty} \varepsilon_k x_k\|^s < \infty$ for all $s > 0$.

PROOF. We begin by taking $u_0 = 0$ in (4.3). Following the proof of Lemma 5.10 [10] we can define $\{\eta_k\}$ and $\{\xi_k\}$ on the same probability space, say Ω , in such a way that $|\eta_k| < |\xi_k|$ a.s.

Let $\{\varepsilon_k\}$ be defined on a probability space Ω_2 and consider the space $\Omega_1 \times \Omega_2$. Let E_1 and E_2 denote expectations with respect to Ω_1 and Ω_2 . Then for $q' = q/r$ and any positive integer n

$$\begin{aligned} E \left[\left\| \sum_{k=1}^n \eta_k x_k \right\|_r^{q'} \right] &= E_1 E_2 \left\| \sum_{k=1}^n \varepsilon_k |\eta_k| x_k \right\|_r^{q'} = E_1 E_2 \left\| \sum_{k=1}^n \varepsilon_k \frac{|\eta_k|}{|\xi_k|} |\xi_k| x_k \right\|_r^{q'} \\ &\leq C_{r,q'} E_1 E_2 \left\| \sum_{k=1}^n \varepsilon_k |\xi_k| x_k \right\|_r^{q'} = C_{r,q'} E \left\| \sum_{k=1}^n \xi_k x_k \right\|_r^{q'} \end{aligned}$$

where we use Lemma 4.3 to obtain the inequality. By assumption $\sum \xi_k x_k$ converges a.s. and the sum is in $L^q(E)$. Therefore by Theorem 4.2 the series converges also in $L^q(E)$ and by completeness of E the above inequality gives the $L^q(E)$ convergence of $\sum \eta_k x_k$ and thus (4.4). The almost sure convergence of $\sum \eta_k x_k$ is now an immediate corollary to the well-known fact that in any metrizable group the convergence in probability of a series with independent summands implies its a.s. convergence.

Now, we shall prove the convergence of $\sum \varepsilon_k x_k$. Let a and b be the constants given in (4.2). Define $\xi'_n = \xi_n I(|\xi_n| < a)$ and $\xi''_n = \xi_n - \xi'_n$, $n = 1, 2, \dots$. Because $P(|\xi'_n| > u) < P(|\xi_n| > u)$ for any $u > 0$, by the case of the theorem already proved both $\sum \xi'_n x_n$ and $\sum \xi''_n x_n$ converges a.s. Define independent $\delta_n = \pm a$ with probability $b/2$ and $\delta_n = 0$ with probability $1 - b$. Since ξ_n are uniformly nondegenerate we have $P(|\delta_n| > u) < P(|\xi''_n| > u)$ for any $u > 0$ so that applying the above reasoning once more we obtain the a.s. convergence of $\sum \delta_n x_n$. It now follows from the proof of Theorem 5.7 [7] that $\sum_{k=1}^{\infty} \varepsilon_k x_k$ converges a.s. (Note that this stage of the proof has nothing to do with a norm and works in a quasi-normed space. In [10], the proof of this stage for E a Banach space makes use of the convexity of the norm, a property not shared by $\|\cdot\|$.)

Once we know that $\sum_{k=1}^{\infty} \varepsilon_k x_k$ converges a.s. we have

$$E \left\| \sum_{k=1}^{\infty} \varepsilon_k x_k \right\|_r^s < \infty \quad (4.5)$$

for all $s > 0$. This follows from Theorem 4.2 since $E[\sup_k \|\varepsilon_k x_k\|_r^s] < \infty$.

Define $\rho_k = u_0 [-u_0]$ if $0 < \eta_k < u_0$ [$-u_0 < \eta_k < 0$] and 0 otherwise and $\rho'_k = \eta_k I(|\eta_k| > u_0)$. Then $P(|\eta_k| > u) < P(|\rho_k + \rho'_k| > u)$ for all $u > 0$ and it follows from the case $u_0 = 0$ that

$$E \left\| \sum_{k=1}^{\infty} \eta_k x_k \right\|_r^{q'} < C_{r,q'} \left[E \left\| \sum_{k=1}^{\infty} \rho_k x_k \right\|_r^{q'} + E \left\| \sum_{k=1}^{\infty} \rho'_k x_k \right\|_r^{q'} \right]. \quad (4.6)$$

We have $E \left\| \sum_{k=1}^{\infty} \rho_k x_k \right\|_r^{q'} < |u_0|^{q'} E \left\| \sum_{k=1}^{\infty} \varepsilon_k x_k \right\|_r^{q'}$. This inequality follows from the fact that $|\rho_k/u_0| < 1$, Lemma 4.3 and (4.5). Also since $P(|\rho'_k| > u) < P(|\xi_k| > u)$ we again use the case $u_0 = 0$ to obtain

$$E \left\| \sum_{k=1}^{\infty} \rho'_k x_k \right\|_r^{q'} < C_{r,q'} E \left\| \sum_{k=1}^{\infty} \xi_k x_k \right\|_r^{q'}.$$

This completes the proof of the theorem, since the a.s. convergence of the series $\sum \eta_k x_k$ obtains in exactly the same way as in the case $u_0 = 0$.

5. Domains of attraction of stable laws with discrete spectral measures. This section is devoted to the proof of Theorem 5.1 which shows the essential relationship between the central limit theorem and spaces of stable type p . However, some of the preliminary results such as Lemma 5.3 and Theorem 5.4 apply to all quasi-normed spaces and are of independent interest.

THEOREM 5.1. *Let E be a quasi-normed linear space, $0 < p < 2$. For the following properties of E :*

- (i) *E is of Rademacher type $p + \varepsilon$, for some $\varepsilon > 0$;*
- (ii) *for all random vectors X with values in E ,*

$$\lim_{a \rightarrow \infty} a^p P(\|X\| > a) = 0 \quad \text{iff} \quad \mathcal{L} \left(\frac{X_1 + \cdots + X_n}{n^{1/p}} \right) \rightarrow \delta_0 \quad (5.1)$$

weakly as $n \rightarrow \infty$, where X_1, X_2, \dots are independent copies of X and δ_0 is the probability measure concentrated at the origin;

- (iii) *Let η_1, η_2, \dots be independent, symmetric, real valued random variables, θ a real valued stable random variable with characteristic function $\exp(-|t|^p)$, $\theta_1, \theta_2, \dots$, independent copies of θ and $c_p = [\int_0^\infty ((\sin x)/x^p) dx]^{-1}$.*

Then given any sequence x_1, x_2, \dots of elements of E such that $Y = \sum_{i=1}^\infty \eta_i x_i$ exists,

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i \in A} \eta_i x_i \right\| > a \right] = c_p \sum_{i \in A} \|x_i\|^p < \infty \quad (5.2)$$

for all subsets A of the positive integers, if and only if $Z = \sum_{i=1}^\infty \theta_i x_i$ exists, and Y satisfies

$$\mathcal{L}((Y_1 + \cdots + Y_n)/n^{1/p}) \rightarrow \mathcal{L}(Z) \quad (5.3)$$

weakly as $n \rightarrow \infty$, where Y_1, Y_2, \dots are independent copies of Y ;

(iv) E is stable type p ;

we have the following implications: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv). Furthermore if E is a Banach space, or if $E = l_r$, $0 < r < 1$, then (iv) \Rightarrow (i), i.e., all of the above properties are equivalent.

Before proving Theorem 5.1 we obtain a number of preliminary results. The first is a relationship between the tail behaviour of the probability distribution of a symmetric random variable and the behaviour of its characteristic function near zero. It is a special case of an estimate in [4, p. 41].

LEMMA 5.2. Let η be a symmetric real valued random variable with distribution function F and characteristic function Φ . Then

$$P\left[|\eta| > \frac{1}{t}\right] < \frac{7}{t} \int_0^t [1 - \Phi(s)] ds. \quad (5.4)$$

Let η_1, η_2, \dots , and $\theta_1, \theta_2, \dots$ be defined as in Theorem 5.1(iii). Recall that a real valued symmetric random variable η is said to be in the domain of normal attraction of θ if $\mathcal{L}(n^{-1/p}(\eta_1 + \cdots + \eta_n)) \rightarrow \mathcal{L}(\theta)$, $n \rightarrow \infty$. A necessary and sufficient condition for η to be in the domain of normal attraction of θ is

$$\lim_{t \rightarrow 0} \frac{1 - \Phi(t)}{|t|^p} = 1. \quad (5.5)$$

Therefore given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$e^{-(1+\varepsilon)|t|^p} < \Phi(t) < e^{-(1-\varepsilon)|t|^p} \quad (5.6)$$

for $|t| < \delta$. Also by Theorem 1 [27] we have

$$\lim_{t \rightarrow \infty} t^p P[|\eta| > t] = c_p. \quad (5.7)$$

Let x_1, x_2, \dots be elements of E . Assume that $Y = \sum_{i=1}^{\infty} \eta_i x_i$ exists. Let c_1, c_2, \dots be an increasing sequence of positive real numbers and define $\eta_i = \eta'_i + \eta''_i$ where

$$\eta''_i = \begin{cases} 0, & |\eta_i| < c_i, \\ \eta_i, & \text{otherwise.} \end{cases} \quad (5.8)$$

Similarly we write $Y = Y' + Y''$ where $Y'' = \sum_{i=1}^{\infty} \eta''_i x_i$.

Note that Y' and Y'' both converge a.s. This follows by a standard comparison theorem argument. Let $\{\eta_i\}$ be defined on the probability space (Ω, \mathcal{F}, P) and let $\{\varepsilon_i\}$ be a Rademacher sequence defined on the probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Consider $\sum \varepsilon_i \eta_i x_i$ on $(\Omega \times \Omega_1, \mathcal{F} \times \mathcal{F}_1, P \times P_1)$. It is

stochastically equivalent to $\sum \eta_i x_i$ and therefore converges a.s. $(P \times P_1)$. By Fubini's theorem for almost all $\omega \in \Omega(P)$, $\sum \varepsilon_i \eta_i(\omega) x_i$ converges a.s. (P_1) . Since $|\eta'_i(\omega)/\eta_i(\omega)| < 1$ we have by Lemma 4.3 that $\sum \varepsilon_i \eta'_i(\omega) x_i$ converges a.s. P_1 for almost all $\omega \in \Omega(P)$. Another application of Fubini's theorem shows that $\sum \varepsilon_i \eta'_i x_i$ converges a.s. $(P \times P_1)$ and therefore $\sum \eta'_i x_i$ converges a.s. (P) . A similar argument shows that Y'' converges a.s. (P) .

Y' will be called the truncated part of Y , even though Y' is not necessarily bounded. Clearly Y' depends on c_1, c_2, \dots . When we speak of all truncated parts Y' of Y , we mean all the Y' terms associated with different admissible sequences c_1, c_2, \dots . Note that in general c_1, c_2, \dots will not be a bounded sequence.

Suppose that $Z = \sum_{i=1}^{\infty} \theta_i x_i$ exists. Then Z is a stable random vector with values in E . We raise the following question. Suppose that each η_i is individually in the domain of attraction of θ ; then under what conditions is Y in the domain of normal attraction of Z ? That is, does

$$\mathcal{L}((Y_1 + \dots + Y_n)/n^{1/p}) \rightarrow \mathcal{L}(Z), \quad n \rightarrow \infty, \quad (5.9)$$

where Y_1, Y_2, \dots , are independent copies of Y ? The next lemma provides an answer to this question. It appears rather abstract but it will prove to be very useful.

LEMMA 5.3. *Let $E, \theta, \{\theta_i\}, \{\eta_i\}, \{x_i\}, Y$, and Z be as above, where each η_i is in the domain of normal attraction of θ . A necessary and sufficient condition for Y to be in the domain of normal attraction of Z is that all truncations Y' of Y are in the domain of normal attraction of δ_0 .*

PROOF. We first prove sufficiency. By (5.7), given $\varepsilon > 0$, for each η_i there exists a $c_i > 0$ such that

$$c_p(1 - \varepsilon)/t^p < P[|\eta_i| > t] < c_p(1 + \varepsilon)/t^p \quad (5.10)$$

for $t > c_i$. Based on the sequence c_1, c_2, \dots we form the decomposition $Y = Y' + Y''$, where $Y'' = \sum_{i=1}^{\infty} \eta''_i x_i$ for $\{\eta''_i\}$ as defined in (5.8). Let Ψ_i be the characteristic function of η''_i . Then

$$1 - \Psi_i(s) = 2 \int_{c_i}^{\infty} \left(\sin \frac{\lambda s}{2} \right)^2 dF_i(\lambda)$$

where $F_i(\lambda)$ is the distribution function of η_i . Therefore we have

$$1 - \Psi_i(s) \leq c_p(1 + \varepsilon)/c_i^p. \quad (5.11)$$

Furthermore if $1/s > c_i$, $1 - \Psi_i(s) \leq -\frac{1}{2} \int_{c_i}^{1/s} \lambda^2 s^2 dH_i(\lambda) + 2 \int_{1/s}^{\infty} dF_i(\lambda)$ where $H_i = 1 - F_i$, and

$$\begin{aligned}
-\frac{1}{2} \int_{c_i}^{1/s} \lambda^2 s^2 dH_i(\lambda) &< \left(\frac{c_p}{2}\right) \left(\frac{1+\varepsilon}{2}\right) (c_i s)^{2-p} s^p + \int_{c_i}^{1/s} s^2 \lambda H(\lambda) d\lambda \\
&< \left(\frac{c_p}{2}\right) (1+\varepsilon) \left[\frac{1}{2} + \frac{1}{2-p}\right] s^p < C'_p s^p
\end{aligned}$$

for some constant C'_p . Also $2 \int_{1/s}^{\infty} dF_i(\lambda) < c_p (1+\varepsilon) s^p$. Therefore, for $1/s > c_i$, $1 - \Psi_i(s) < C_p s^p$, where C_p is a constant independent of i . Combining this with (5.11) we get $1 - \Psi_i(s) < C_p s^p$ for all $s > 0$. Let δ_p be such that $(1 - C_p s^p/n)^n > 1 - 2C_p s^p$ for $|s| < \delta_p$. (Clearly $\delta_p > 0$.) Thus for $|s| < \delta_p$

$$\Psi_i^n(s/n^{1/p}) > (1 - C_p s^p/n)^n > 1 - 2C_p s^p,$$

and

$$1 - \Psi_i^n(s/n^{1/p}) < 2C_p s^p. \quad (5.12)$$

By (5.12) and Lemma 5.2, $u_0 = 1/\delta_p$ and $u > u_0$. We get

$$P \left[\left| \frac{\eta_{i,1}'' + \cdots + \eta_{i,n}''}{n^{1/p}} \right| > u \right] < Cu \int_0^{1/u} 2C_p s^p ds < \frac{K_p c_p}{u^p} \quad (5.13)$$

for some constant K_p independent of i . This is the estimate that we will use in the next step of the proof.

By hypothesis $\sum_{i=1}^{\infty} \theta_i x_i$ exists. Since this implies that $\lim_{i \rightarrow \infty} \|\theta_i x_i\| = 0$ a.s., it follows from the Borel-Cantelli lemma that $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$ (cf. also (1.3)). Following §4 let $N = \sup_i \|\theta_i x_i\|_r$. Then

$$\begin{aligned}
P(N > a^r) &< \sum_{i=1}^{\infty} P[|\theta_i| \|x_i\| > a] \\
&< \frac{2(1+\varepsilon) \sum_{i=1}^{\infty} \|x_i\|^p}{a^p}.
\end{aligned} \quad (5.14)$$

Thus $E(N^{q/r}) < \infty$ for $q < p$ and by Theorem 4.2 we also have

$$E \left\| \sum_{i=1}^{\infty} \theta_i x_i \right\|^q < \infty, \quad q < p \quad (5.15)$$

(as we might expect from [8]).

We use Theorem 4.4 together with (5.13) to obtain the essential inequality of this proof, that for any N, n and $q < p \wedge r$

$$\begin{aligned}
&E \left\| \sum_{k=N}^{\infty} \left(\frac{\eta_{k,1}'' + \cdots + \eta_{k,n}''}{n^{1/p}} \right) x_k \right\|^q \\
&< C'_1 |u_0|^q E \left\| \sum_{k=N}^{\infty} \varepsilon_k x_k \right\|^q + C'_2 E \left\| \sum_{k=N}^{\infty} \theta_k x_k \right\|^q.
\end{aligned} \quad (5.16)$$

(Note that in using Theorem 4.4 we compare the random variables $(\eta''_{k,1} + \dots + \eta''_{k,n})/n^{1/p}$ with $2K_p\theta_p$ in order to satisfy (4.3). The factor $2K_p$ is then absorbed into the constant C'_2 . The constants C'_1 and C'_2 depend upon p , q , r and K_p ; however all these terms remain fixed throughout what follows.) The existence of the first term on the right is asserted in Theorem 4.4, of the second term on the right in (5.15). It also follows from Theorem 4.2 that both of these terms can be made arbitrarily small by taking N sufficiently large.

We now show that Y'' is in the domain of normal attraction of Z . Since bounded Lipschitz functions of order $\alpha < 1$ form an algebra of functions on E which separate points of E by Lemma 3.1 [9] it is sufficient to show that

$$\lim_{n \rightarrow \infty} Ef \left(\sum_{i=1}^{\infty} \frac{\eta''_{i,1} + \dots + \eta''_{i,n}}{n^{1/p}} x_i \right) = Ef \left(\sum_{i=1}^{\infty} \theta_i x_i \right)$$

for all bounded functions f satisfying $|f(x) - f(y)| < K\|x - y\|^\alpha$ for all $x, y \in E$ and some $\alpha < 1$ (K is a constant depending on f). We have

$$\begin{aligned} & \left| Ef \left(\sum_{i=1}^{\infty} \frac{\eta''_{i,1} + \dots + \eta''_{i,n}}{n^{1/p}} x_i \right) - Ef \left(\sum_{i=1}^{\infty} \theta_i x_i \right) \right| \\ & \leq \left| Ef \left(\sum_{i=1}^{\infty} \frac{\eta''_{i,1} + \dots + \eta''_{i,n}}{n^{1/p}} x_i \right) - Ef \left(\sum_{i=1}^{N-1} \frac{\eta''_{i,1} + \dots + \eta''_{i,n}}{n^{1/p}} x_i \right) \right| \\ & \quad + \left| Ef \left(\sum_{i=1}^{N-1} \frac{\eta''_{i,1} + \dots + \eta''_{i,n}}{n^{1/p}} x_i \right) - Ef \left(\sum_{i=1}^{N-1} \theta_i x_i \right) \right| \\ & \quad + \left| Ef \left(\sum_{i=1}^{N-1} \theta_i x_i \right) - Ef \left(\sum_{i=1}^{\infty} \theta_i x_i \right) \right|. \end{aligned} \quad (5.17)$$

Let the index α satisfy $0 < \alpha < p \wedge r \wedge 1$. The first term to the right of the inequality in (5.17) is less than or equal to

$$E \left\| \sum_{i=N}^{\infty} \frac{\eta''_{i,1} + \dots + \eta''_{i,n}}{n^{1/p}} x_i \right\|^\alpha \quad (5.18)$$

and the third term is less than or equal to $E \|\sum_{i=N}^{\infty} \theta_i x_i\|^\alpha$. It follows from (5.16) and the remarks following (5.16) that each of these terms can be made arbitrarily small by taking N sufficiently large. On the other hand, as n goes to infinity (for fixed N) the second term to the right of the inequality in (5.17) goes to zero. This is simply a finite dimensional result and each η''_i is in the domain of attraction of θ .

Thus Y'' is in the domain of attraction of Z and by hypothesis Y' is in the domain of attraction of δ_0 . It follows from Theorem 4.1 [34, p. 25] that Y is

in the domain of attraction of Z . This completes the proof of sufficiency.

In proving necessity we begin with the assumption that Y is in the domain of normal attraction of Z . We write $Y = Y' + Y''$ where Y' is any truncation of Y . Let S'_n and S''_n be defined as above with respect to this truncation. It follows from the symmetry of the η_i that $S'_n + S''_n$ and $S'_n - S''_n$ are equally distributed random vectors. Therefore, for any absolutely r -convex set K , $P[S'_n + S''_n \notin K] \geq \frac{1}{2} P[S'_n \notin 2^{1/r-1}K]$. Since the measures corresponding to $S'_n + S''_n$ are uniformly tight and since the closure of the absolute r -convexification of a compact set in E is also compact, we see that the measures corresponding to S'_n are also uniformly tight. Now, since $\sum_{i=1}^{\infty} \eta_i x_i$ is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x_i$, for any linear functional $x^* \in E^*$ we have that $\sum_{i=1}^{\infty} \eta_i x^* x_i$ is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x^* x_i$. We will show that this implies that $\sum_{i=1}^{\infty} \eta'_i x^* x_i$ is in the domain of normal attraction of δ_0 . Since this is true for all x^* and since we know that laws of S'_n are uniformly tight we have that S'_n is in the domain of normal attraction of δ_0 .

Let $a_i = x^* x_i$. Thus a_i is a real number. We now show that if $\sum_{i=1}^{\infty} \eta_i a_i$ is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i a_i$ then $\sum_{i=1}^{\infty} \eta'_i a_i$ is in the domain of normal attraction of δ_0 , where δ_0 is the unit mass at zero of the real line. Considering the relevant characteristic functions we see that, for all integers $N > 1$, $\sum_{i=N}^{\infty} \eta_i a_i$ is in the domain of normal attraction of $\sum_{i=N}^{\infty} \theta_i a_i$. Therefore

$$\lim_{a \rightarrow \infty} a^p P \left[\left| \sum_{i=N}^{\infty} \eta_i a_i \right| > a \right] = \frac{c_p \sum_{i=N}^{\infty} |\eta_i a_i|^p}{a^p}.$$

(This is a classical result. It can be readily derived using (5.5) and Theorem 1 [27] taking into account the fact that $\sum_{i=N}^{\infty} \theta_i a_i$ is equal in distribution to $(\sum_{i=N}^{\infty} |\eta_i a_i|^p)^{1/p} \theta$.)

Consider $\sum_{i=1}^{\infty} \eta'_i a_i$. For each N , there exists an M_N such that $|\sum_{i=1}^N \eta'_i a_i| < M_N$. Therefore

$$\begin{aligned} P \left[\left| \sum_{i=1}^{\infty} \eta'_i a_i \right| > a \right] &< P \left[\left| \sum_{i=1}^{N-1} \eta'_i a_i \right| > M_N \right] + P \left[\left| \sum_{i=N}^{\infty} \eta'_i a_i \right| > -M_N \right] \\ &= P \left[\left| \sum_{i=N}^{\infty} \eta'_i a_i \right| > a - M_N \right]. \end{aligned}$$

Note that $\sum \eta'_i a_i + \sum \eta''_i a_i$ and $\sum \eta'_i a_i - \sum \eta''_i a_i$ are equally distributed; therefore

$$\begin{aligned} P \left[\left| \sum_{i=N}^{\infty} \eta'_i a_i \right| > a - M_N \right] &< 2P \left[\left| \sum_{i=N}^{\infty} \eta_i x_i \right| > a - M_N \right] \\ &< \frac{2c_p \sum_{i=N}^{\infty} |\eta_i a_i|^p}{a^p (1 - M_N/a)^p}. \end{aligned}$$

Thus we have $\lim_{a \rightarrow \infty} a^p P[|\sum_{i=1}^{\infty} \eta'_i a_i| > a] \leq K \sum_{i=1}^{\infty} |a_i|^p$ for some constant K and since this holds for all N we get $\lim_{a \rightarrow \infty} a^p P[|\sum_{i=1}^{\infty} \eta'_i a_i| > a] = 0$. Since the real line is of Rademacher type 2 we can use Theorem 3.1 to show that $\sum_{i=1}^{\infty} \eta'_i a_i$ is in the domain of normal attraction of δ'_0 . This completes the proof of Lemma 5.3.

The next result provides the "only if" part of Theorem 5.1(iii); however since it applies to all quasi-normed spaces we state it as a separate theorem.

THEOREM 5.4. *Let θ , $\{\theta_i\}$, $\{x_i\}$ be as above and let $\{\eta_i\}$ be a sequence of independent symmetric real valued random variables. Assume that $\sum \theta_i x_i$ exists, $\sum \eta_i x_i$ exists and $\sum \eta_i x_i$ is in the domain of normal attraction of $\sum \theta_i x_i$. Then*

$$\lim_{a \rightarrow \infty} a^p P\left[\left\|\sum_{i=1}^{\infty} \eta_i x_i\right\| > a\right] = c_p \sum_{i=1}^{\infty} \|x_i\|^p. \quad (5.19)$$

Note. This result holds for $\eta_i = \theta_i$; also $\sum \theta_i x_i$ exists implies $\sum \|x_i\|^p < \infty$.

PROOF. By (1.9) we can obtain a linear functional $x^* \in E^*$ such that $x^*(\sum \eta_i x_i) = \sum \eta_i x^* x_i$ and similarly $x^*(\sum \theta_i x_i) = \sum \theta_i x^* x_i$. Thus η_i is in the domain of normal attraction of θ .

Since $Y = \sum_{i=1}^{\infty} \eta_i x_i$ is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x_i$ it follows from Lemma 5.3 that all truncations $Y' = \sum_{i=1}^{\infty} \eta'_i x_i$ of Y are in the domain of normal attraction of δ_0 . By (3.2) this implies that

$$\lim_{a \rightarrow \infty} a^p P[\|Y'\| > a] = 0 \quad (5.20)$$

for all truncations Y' of Y . For a given $\varepsilon > 0$ we choose the sequence $\{c_i\}$ established at the beginning of the proof of Lemma 5.3 and form the corresponding decomposition $Y = Y' + Y''$, where $Y'' = \sum_{i=1}^{\infty} \eta''_i x_i$ for η''_i as defined in (5.8). Following Lemma 4.1 define $F(t) = P(\|\sum \eta''_i x_i\|_r > t)$, $N = \sup_i \|\eta''_i x_i\|_r$ and $G(t) = P(N > t)$. By Lemma 4.1 we have

$$F(3t) \leq G(t) + 4F^2(t) \quad (5.21)$$

and by (5.10)

$$G(t) \leq \sum_{i=1}^{\infty} P(\|\eta''_i x_i\|_r > t) \leq \frac{c_p (1 + \varepsilon) \sum_{i=1}^{\infty} \|x_i\|^p}{t^{p/r}}. \quad (5.22)$$

We will now show that there exists a t_0 such that

$$F(t) \leq \frac{C \sum_{i=1}^{\infty} \|x_i\|^p}{t^{p/r}} \quad (5.23)$$

for $t > t_0$ and for some fixed constant C . If $4F^2(t) > \frac{1}{2} F(3t)$ for all $t > t_1$ for some fixed $t_1 > 0$, then there exist numbers $\alpha > 0$, $\beta > 0$, $t_2 > 0$ such that for $t > t_2$, $F(t) < e^{-\alpha|t|^{\beta}}$ and this implies (5.23). Therefore we will assume that there exists a sequence $\{t_k\}$, $t_k \uparrow \infty$ such that $4F^2(t_k) \leq \frac{1}{2} F(3t_k)$. By (5.21)

and (5.22) we have

$$F(3t_k) \leq 2G(t_k) \leq \frac{2c_p(1+\varepsilon)\sum_{i=1}^{\infty}\|x_i\|^p}{t_k^{p/r}},$$

$$F(9t_k) \leq G(3t_k) + 4F^2(3t_k)$$

$$\leq \frac{c_p(1+\varepsilon)\sum_{i=1}^{\infty}\|x_i\|^p}{(3t_k)^{p/r}} + \frac{16(c_p)^2(1+\varepsilon)^2(\sum_{i=1}^{\infty}\|x_i\|^p)^2}{t_k^{2p/r}}. \quad (5.24)$$

Therefore for some t_{k_0} sufficiently large we have for $t_k \geq t_{k_0}$ that

$$F(9t_k) \leq \frac{2c_p(1+\varepsilon)\sum_{i=1}^{\infty}\|x_i\|^p}{(3t_k)^{p/r}}.$$

Thus

$$F(3^j t_k) \leq \frac{2c_p(1+\varepsilon)\sum_{i=1}^{\infty}\|x_i\|^p}{(3^{j-1}t_k)^{p/r}}. \quad (5.25)$$

Interpolating we get (5.23). We now sharpen the constant in (5.23). Using (5.22) and (5.23) in Lemma 4.1 with $s = (k-2)t$ we get

$$F(kt) \leq \frac{c_p(1+\varepsilon)\sum_{i=1}^{\infty}\|x_i\|^p}{(kt)^{p/r}} \left[\left(\frac{k}{k-2} \right)^{p/r} + \frac{4C^2\sum_{i=1}^{\infty}\|x_i\|^p}{c_p(1+\varepsilon)} \left(\frac{k}{t} \right)^{p/r} \right] \quad (5.26)$$

for $t \geq t_0$ and all integers $k \geq 3$. Let $s_0 > t_0 > 10^4$ and $s_0^{5/4} + 1 \leq s \leq s_0^{7/4}$. Then there exists a $t \geq s_0$ and an integer k , $s_0^{1/4} \leq k \leq s_0^{3/4}$, such that $s = kt$. By (5.26) we have

$$F(s) \leq \frac{c_p(1+\varepsilon)\sum_{i=1}^{\infty}\|x_i\|^p}{s^{p/r}} \left[1 + \frac{4p}{rs_0^{p/4}} + \frac{4C^2\sum_{i=1}^{\infty}\|x_i\|^p}{c_p(1+\varepsilon)} \left(\frac{1}{s_0} \right)^{p/4r} \right].$$

Note that as s_0 increases the bound on the right decreases. Thus we get

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i=1}^{\infty} \eta_i'' x_i \right\| > a \right] \leq c_p(1+\varepsilon) \sum_{i=1}^{\infty} \|x_i\|^p. \quad (5.27)$$

The value of ε depends on the sequence $\{c_i\}$. Increasing all the $\{c_i\}$ allows us to decrease ε without affecting (5.20).

The upper bound in (5.19) is obtained by the following argument:

$$P[\|Y\|_r > a^r] = P[\|Y' + Y''\|_r > a^r]$$

$$\leq P[\|Y''\|_r > (a/(1+\delta))^r] + P[\|Y'\|_r > a^r(r\delta)]$$

for some small $\delta > 0$. Thus

$$\begin{aligned} \lim_{a \rightarrow \infty} a^p P[\|Y\| > a] &\leq \lim_{a \rightarrow \infty} a^p P[\|Y'\| > a/(1 + \delta)] \\ &\quad + \lim_{a \rightarrow \infty} a^p P[\|Y'\| > a(r\delta)^{1/r}] \\ &\leq 2(1 + \epsilon) \sum_{i=1}^{\infty} \|x_i\|^p. \end{aligned} \quad (5.28)$$

Since, with $\beta = (r\delta)^{1/r}$,

$$\lim_{a \rightarrow \infty} a^p P[\|Y'\| > a\beta] = \beta^{-p} \lim_{a \rightarrow \infty} a^p P[\|Y'\| > a] = 0.$$

Since (5.28) holds for all $\epsilon > 0$ we obtain the upper bound in (5.19).

Obtaining the lower bound is easier. For a fixed value of N , to be specified below, let $a_0 > \sup_{1 \leq i \leq N} c_i(1 + \delta)\|x_i\|^{-1}$. Then for $\delta > 0$ sufficiently small and $a > a_0$

$$\begin{aligned} &P\left[\left\|\sum_{i=1}^N \eta_i'' x_i\right\|_r > (a(1 + \delta))^r\right] \\ &> \sum_{i=1}^N \left\{ P[\|\eta_i'' x_i\|_r > [a(1 + \delta)]^r] \right. \\ &\quad \times P\left[\left\|\sum_{\substack{j=1 \\ j \neq i}}^N \eta_j x_j\right\|_r < r\delta a^r, \sup_{\substack{1 \leq j \leq N \\ j \neq i}} \|\eta_j'' x_j\|_r < [a(1 + \delta)]^r\right] \Big\} \\ &> \sum_{i=1}^N \frac{c_p(1 - \epsilon)\|x_i\|^p}{(1 + \delta)a^p} \left\{ 1 - P\left[\left\|\sum_{\substack{j=1 \\ j \neq i}}^N \eta_j'' x_j\right\|_r > r\delta a^r\right] \right. \\ &\quad \left. - \sum_{\substack{j=1 \\ j \neq i}}^N P[\|\eta_j'' x_j\|_r > [a(1 + \delta)]^r] \right\}. \end{aligned}$$

By (5.27) we have

$$P\left[\left\|\sum_{\substack{j=1 \\ j \neq i}}^N \eta_j'' x_j\right\|_r > r\delta a^r\right] < \left(\frac{1}{r\delta}\right)^{1/r} \frac{c_p(1 + \epsilon)\sum_{j=1}^N \|x_j\|^p}{a^p}.$$

Also

$$\sum_{\substack{j=1 \\ j \neq i}}^N P[\|\eta_j'' x_j\|_r > [a(1 + \delta)]^r] < \frac{c_p(1 + \epsilon)\sum_{j=1}^N \|x_j\|^p}{(1 + \delta)^p a^p}.$$

It follows that

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i=1}^N \eta_i'' x_i \right\| > a \right] \geq c_p (1 - \varepsilon) \sum_{i=1}^N \|x_i\|^p. \quad (5.29)$$

Next we note that for a sufficiently large

$$\begin{aligned} & P \left[\left\| \sum_{i=1}^{\infty} \eta_i'' x_i \right\|_r > a^r \right] \\ & \geq P \left[\left\| \sum_{i=1}^N \eta_i'' x_i \right\|_r > [(1 + \delta)a]^r \right] - P \left[\left\| \sum_{i=N+1}^{\infty} \eta_i'' x_i \right\|_r > r\delta a^r \right] \\ & \geq \frac{c_p (1 - 2\varepsilon) \sum_{i=1}^N \|x_i\|^p}{(1 + \delta)^p a^p} - \frac{c_p \sum_{i=N+1}^{\infty} \|x_i\|^p}{(r\delta)^{1/r} a^p}. \end{aligned} \quad (5.30)$$

Now we choose N so that the last term above is negligible with respect to the preceding term. Thus we get

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i=1}^{\infty} \eta_i'' x_i \right\| > a \right] \geq c_p (1 - 4\varepsilon) \sum_{i=1}^N \|x_i\|^p$$

and since this holds for all N

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i=1}^{\infty} \eta_i'' x_i \right\| > a \right] \geq c_p (1 - 4\varepsilon) \sum_{i=1}^{\infty} \|x_i\|^p. \quad (5.31)$$

We repeat the argument of (5.30) except this time we write $Y = Y' + Y''$. Taking (5.20) into account we get

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i=1}^{\infty} \eta_i'' x_i \right\| > a \right] \geq c_p (1 - 8\varepsilon) \sum_{i=1}^{\infty} \|x_i\|^p,$$

and since this holds for all $\varepsilon > 0$ we obtain the lower bound in (5.19).

PROOF OF THEOREM 5.1. (i) \Rightarrow (ii). This is Theorem 3.1.

(i) \Rightarrow (iii). Since E is of Rademacher type $p + \varepsilon$, $\varepsilon > 0$, it is also of stable type p . Therefore we have that $\sum \|x_i\|^p < \infty$ implies $\sum \theta_i x_i$ exists. Also if we take the set A in (5.2) to be a single integer we see that each η_i is individually in the domain of attraction of θ . In view of Lemma 5.3 and (ii) (since (i) also implies (ii)) it suffices to show that condition (5.2) implies that for any truncation $Y' = \sum_{i=1}^{\infty} \eta_i' x_i$ of Y we have

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i=1}^{\infty} \eta_i' x_i \right\| > a \right] = 0. \quad (5.32)$$

Indeed, for each N there exists an M_N such that $\|\sum_{i=1}^{N-1} \eta'_i x_i\|_r < M_N$. Thus

$$\begin{aligned} P \left[\left\| \sum_{i=1}^{\infty} \eta'_i x_i \right\|_r > a^r \right] \\ < P \left[\left\| \sum_{i=1}^N \eta'_i x_i \right\|_r > M_N \right] + P \left[\left\| \sum_{i=N}^{\infty} \eta'_i x_i \right\|_r > a^r - M_N \right] \\ = P \left[\left\| \sum_{i=N}^{\infty} \eta'_i x_i \right\|_r > a^r - M_N \right]. \end{aligned}$$

Note that $\sum \eta'_i x_i + \sum \eta''_i x_i$ and $\sum \eta'_i x_i - \sum \eta''_i x_i$ are equally distributed. Therefore

$$\begin{aligned} P \left[\left\| \sum_{i=N}^{\infty} \eta'_i x_i \right\|_r > a^r - M_N \right] &< 2P \left[\left\| \sum_{i=N}^{\infty} \eta_i x_i \right\|_r > a^r - M_N \right] \\ &< \frac{4c_p \sum_{i=N}^{\infty} \|x_i\|^p}{a^p (1 - M_N/a^r)^{p/r}}. \end{aligned} \quad (5.33)$$

Thus $\lim_{a \rightarrow \infty} a^p P[\|\sum_{i=1}^{\infty} \eta'_i x_i\| > a] < K \sum_{i=N}^{\infty} \|x_i\|^p$ for some constant K and since this holds for all N we get (5.32). This completes the proof of (i) \Rightarrow (iii).

(iii) \Rightarrow (iv). Actually we show not (iv) implies not (iii). If E is not of stable type p there exists a sequence $\{x_i\}$ of elements in E such that $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$ but such that $\sum_{i=1}^{\infty} \theta_i x_i$ does not exist. For such a sequence consider the series $\sum_{i=1}^{\infty} \eta''_i x_i$ where η''_i are independent and

$$\eta''_i = \begin{cases} 0 & \text{if } |\theta_i| < c_i, \\ \theta_i & \text{if } |\theta_i| > c_i. \end{cases} \quad (5.34)$$

Here the sequence c_i is chosen such that $\sum_{i=1}^{\infty} P[|\eta''_i| \neq 0] < \infty$. Thus, by the Borel-Cantelli lemma, $\sum_{i=1}^{\infty} \eta''_i x_i < \infty$ and defines an E -valued random vector. It follows from (5.27) and (5.31) that

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i \in A} \eta''_i x_i \right\| > a \right] = c_p \sum_{i \in A} \|x_i\|^p < \infty, \quad (5.35)$$

since in both (5.27) and (5.31) the indexing of the terms and the values of the $\{x_i\}$ were completely arbitrary except for the assumption $\sum_{i=1}^{\infty} \eta''_i x_i$ exists. Neither of these statements depended upon the existence of $\sum_{i=1}^{\infty} \theta_i x_i$. Therefore $\sum_{i=1}^{\infty} \eta''_i x_i$ satisfies the left side of (5.2) but its law is not in the domain of attraction of any measure, since by checking the finite dimensional distributions we see that the limiting measure would have to have the distribution of $\sum_{i=1}^{\infty} \theta_i x_i$ and this series does not converge.

(ii) \Rightarrow (iv). Again we actually show not (iv) implies not (ii). We will find a counterexample closely related to the one just used. Again we assume that E

is not of stable type p and choose the same sequence $\{x_i\}$ for which $\sum \theta_i x_i$ does not converge a.s. By the extended Ito and Nisio theorem it does not converge in probability either. Therefore we can find an $\varepsilon > 0$ and a sequence $N_k \uparrow \infty$ such that

$$P \left[\left\| \sum_{i=N_k}^{N_{k+1}} \theta_i x_i \right\| > \varepsilon \right] > \varepsilon.$$

Now consider $X^k = \sum_{i=N_k}^{N_{k+1}} \eta_i'' x_i$ for η_i'' as defined in (5.34). Each η_i'' is individually in the domain of attraction of θ_i and since we are only considering a finite sum we can use characteristic functionals to see that

$$\mathcal{L} \left(\frac{X_1^k + \dots + X_m^k}{m^{1/p}} \right) \rightarrow \mathcal{L} \left(\sum_{i=N_k}^{N_{k+1}} \theta_i x_i \right)$$

where X_1^k, X_2^k, \dots are independent copies of X^k . It follows that there exists an m_k such that

$$P \left[\left\| \sum_{i=N_k}^{N_{k+1}} \frac{\eta_{i,1}'' + \dots + \eta_{i,m_k}''}{m_k^{1/p}} x_i \right\| > \frac{\varepsilon}{2} \right] > \frac{\varepsilon}{2}. \quad (5.36)$$

Now for $N_k < i < N_{k+1}$ let

$$\rho_i = \begin{cases} \eta_i'' & \text{if } |\eta_i''| < M_k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\rho_{i,1}, \rho_{i,2}, \dots$ be independent copies of ρ_i . Here we choose M_k such that

$$P[\rho_{i,j} = \eta_{i,j}''; i = N_k, \dots, N_{k+1}, j = 1, \dots, m_k] > 1 - \varepsilon/4. \quad (5.37)$$

Therefore, using (5.36) and (5.37) we get

$$P \left[\left\| \sum_{i=N_k}^{N_{k+1}} \frac{\rho_{i,1} + \dots + \rho_{i,m_k}}{m_k^{1/p}} x_i \right\| > \frac{\varepsilon}{2} \right] > \frac{\varepsilon}{4},$$

and by our standard argument involving the sums of symmetric random variables we have

$$P \left[\left\| \sum_{i=1}^{\infty} \frac{\rho_{i,1} + \dots + \rho_{i,m_k}}{m_k^{1/p}} x_i \right\| > \frac{\varepsilon}{2} \right] > \frac{\varepsilon}{8}. \quad (5.38)$$

Now $\sum_{i=1}^{\infty} \rho_i x_i$ converges a.s. for the same reason that $\sum_{i=1}^{\infty} \eta_i'' x_i$ converges a.s. We see by (5.38) that $\sum_{i=1}^{\infty} \rho_i x_i$ does not satisfy the weak law of large numbers. Also each ρ_i is a truncation of η_i'' and $\sum \eta_i'' x_i$ satisfies (5.35). Therefore the same argument used to show that (5.2) implies (5.32) gives us $\lim_{a \rightarrow \infty} a^p P[\|\sum_{i=1}^{\infty} \rho_i x_i\| > a] = 0$. Thus $\sum_{i=1}^{\infty} \rho_i x_i$ provides the counterexample.

(ii) \Rightarrow (iii). Since (ii) \Rightarrow (iv) we have both (5.1) and the fact that E is of stable type p . The proof of (i) \Rightarrow (iii) uses precisely these two facts. This completes the proof of (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

When E is a Banach space the implication (iv) \Rightarrow (i) is one of the major results of [22]. For l_r , $0 < r < 1$, (iv) \Rightarrow (i) is elementary.

6. Additional results and examples concerning the central limit theorem in quasi-normed spaces. Despite the critical role of spaces of stable type p in Theorem 5.1, when the η_i 's are also identically distributed we have a central limit theorem in all quasi-normed spaces.

THEOREM 6.1. *Let θ be a real valued, symmetric random variable with characteristic function $\exp - |t|^p$, η be a real valued symmetric random variable which is in the domain of normal attraction of θ , and $\{\theta_i\}$ and $\{\eta_i\}$ be independent copies of θ and η . If $\{x_i\} \subset E$ is such that $\sum_{i=1}^{\infty} \theta_i x_i$ exists, then $\sum_{i=1}^{\infty} \eta_i x_i$ exists and is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x_i$.*

PROOF. Since the $\{\eta_i\}$ are identically distributed for every $\varepsilon > 0$ one can find a $c > 0$ such that

$$c_p(1 - \varepsilon)/t^p \leq P[|\eta_i| > t] \leq c_p(1 + \varepsilon)/t^p$$

for all $t \geq c$ (see (5.10)). Comparing η_i to $2\theta_i$ in (4.3) we see that Theorem 4.4 implies that $\sum \eta_i x_i$ converges a.s. Define $\eta_i = \eta'_i + \eta''_i$ with respect to $c_i = c$ as in (5.8). Define Y' and Y'' as in the remarks following (5.8). In the proof of Lemma 5.3 we showed that Y'' is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x_i$. Considering the remarks at the end of Lemma 5.3 we need only show that $Y' = \sum_{i=1}^{\infty} \eta''_i x_i$ is in the domain of normal attraction of δ_0 . Note that η'_i are uniformly bounded.

Let $\{\rho_i\}$ be independent normal random variables with mean zero and variance 1. It follows from Theorem 4.4 that $\sum_{i=1}^{\infty} \rho_i x_i < \infty$. Therefore by Theorem 4.1 [19]

$$\mathcal{L}\left(\frac{Y'_1 + \cdots + Y'_n}{n^{1/2}}\right) \rightarrow \mathcal{L}\left(\sum_{i=1}^{\infty} \rho_i x_i\right) \quad (6.1)$$

where Y'_1, Y'_2, \dots , are independent copies of Y' . (Actually Theorem 4.1 of [19] is proved only for continuous processes in Banach spaces; however the result holds in this case also. This is because (4.17) [19] enables us to use the argument of (5.16) of this paper and what follows to obtain (6.1).) Since $p < 2$ we see that Y' is in the domain of normal attraction of δ_0 and this completes the proof.

EXAMPLE 6.1. At the other extreme of Theorem 6.1 we now show that given any series $\sum_{i=1}^{\infty} \theta_i x_i$ as above in any quasi-normed space E there exists a series $\sum_{i=1}^{\infty} \eta_i x_i$ such that each η_i is individually in the domain of normal attraction

of θ but $\sum_{i=1}^{\infty} \eta_i x_i$ is not in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x_i$ (both series exist).

By Lemma 5.3 it is enough to find a sequence of independent random variables $\{\xi_i\}$ such that $|\xi_i| < c_i$, $\sum_{i=1}^{\infty} \xi_i x_i$ exists but $\sum_{i=1}^{\infty} \xi_i x_i$ is not in the domain of normal attraction of δ_0 . Define

$$\xi_i = \begin{cases} 0 & \text{with probability } 1 - p_i, \\ \pm \frac{i^{1/p}}{\|x_i\|} & \text{each with probability } p_i/2, \end{cases}$$

where $p_i = i^{-3/2}$. If $\sum_{i=1}^{\infty} \xi_i x_i$ is in the domain of normal attraction of δ_0 then

$$\lim_{m \rightarrow \infty} P \left[\left\| \sum_{i=m}^{\infty} \xi_i^{(m)} x_i \right\| > 2^{1-1/r} \right] = 0 \quad (6.2)$$

where $\xi_i^{(m)} = (\xi_{i,1} + \dots + \xi_{i,m})/m^{1/p}$, the $\xi_{i,j}$ being independent copies of ξ_i . Let

$$\gamma_{i,m} = \begin{cases} \xi_{i,j} & \text{if } \xi_{i,j} \neq 0 \text{ and } \xi_{i,k} = 0, k = 1, \dots, j-1, j+1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\gamma_{i,m} = \pm i^{1/p} \|x_i\|^{-1}$ each with probability $\frac{1}{2} m p_i (1 - p_i)^{m-1}$. Define $\gamma'_{i,m} = \xi_i^{(m)} m^{1/p} - \gamma_{i,m}$. Note that $\gamma'_{i,m} + \gamma_{i,m}$ has the same distribution as $\gamma'_{i,m} - \gamma_{i,m}$. Therefore

$$P[|\xi_i^{(m)} m^{1/p}| > 2^{1-1/r}] \geq \frac{1}{2} P[|\gamma_{i,m}| > 1]$$

or equivalently

$$P[\|\xi_i^{(m)}\| > 2^{1-1/r}] \geq \frac{1}{2} P[\|(\gamma_{i,m}/m^{1/p})x_i\| > 1].$$

Therefore

$$\begin{aligned} P \left[\sup_{m \leq i} \|\xi_i^{(m)} x_i\| < 2^{1-1/r} \right] &= \prod_{i=m}^{\infty} [1 - P(\|\xi_i^{(m)} x_i\| > 2^{1-1/r})] \\ &< \prod_{i=m}^{\infty} \left[1 - \frac{1}{2} P \left[\left\| \frac{\gamma_{i,m}}{m^{1/p}} x_i \right\| > 1 \right] \right]. \end{aligned}$$

For $i \geq m$, $\|(\gamma_{i,m}/m^{1/p})x_i\| = i^{1/p}/m^{1/p} \geq 1$ with probability $m p_i (1 - p_i)^{m-1} > m p_i / 2$ for $i, m \geq 10$. Therefore

$$P \left[\sup_i \|\xi_i^{(m)} x_i\| < 2^{1-1/r} \right] < \exp \left(- \frac{1}{4} \sum_{i=m}^{\infty} \frac{m}{i^{3/2}} \right) < \frac{1}{2} \quad (6.3)$$

for m sufficiently large. Now by symmetry and independence

$$P \left[\left\| \sum_{i=m}^{\infty} \xi_i^{(m)} x_i \right\| > 2^{1-1/r} \right] \geq \frac{1}{2} P \left[\sup_{m \leq i} \|\xi_i^{(m)} x_i\| \geq 2^{1-1/r} \right]$$

and using (6.3) we get

$$P \left[\left\| \sum_{i=m}^{\infty} \xi_i^{(m)} x_i \right\| > 2^{1-1/r} \right] > \frac{1}{4}$$

which contradicts (6.2).

We have been studying series of the form $Y = \sum_{i=1}^{\infty} \eta_i x_i$ where $x_i \in E$, E a quasi-normed space and the $\{\eta_i\}$ are independent, symmetric, real valued random variables such that each η_i is in the domain of attraction of θ , a stable random variable with characteristic function $\exp -|t|^p$, $0 < p < 2$. We have investigated the question of whether $(n^{-1/p}(Y_1 + \dots + Y_n))$ converges weakly. It is clear that if it does then the limit measure must be the law of $\sum_{i=1}^{\infty} \theta_i x_i$ where $\{\theta_i\}$ are independent copies of θ . If $\sum_{i=1}^{\infty} \theta_i x_i$ does not exist (i.e., the partial sums do not converge a.s.) then $\mathcal{L}(n^{-1/p}(Y_1 + \dots + Y_n))$ can not converge weakly. Adopting the terminology used in the study of the conventional central limit theorem we say that $\sum_{i=1}^{\infty} \eta_i x_i$ is not prestable if $\sum_{i=1}^{\infty} \theta_i x_i$ does not exist. The counterexamples used in Theorem 5.1 to show not (iv) implies not (ii) and not (iv) implies not (iii) were both of this kind, i.e., the measures introduced were not prestable. In our final example we present prestable measures that satisfy hypothesis (5.2) of Theorem 5.1, but which do not satisfy the central limit theorem.

EXAMPLE 6.2. Consider the space l^q with the usual q -homogeneous norm $\|z\|_q = \sum_{i=1}^{\infty} |z_i|^q$ for $z = (z_1, z_2, \dots) \in l^q$ and quasi-norm $\|z\| = \|z\|_q^{1/q}$. We show the following: for $0 < q < p < 2$, there exists $x_i \in l^q$, $i = 1, 2, \dots$, such that $\sum_{i=1}^{\infty} \eta_i x_i$ exists in l^q , $Z = \sum_{i=1}^{\infty} \theta_i x_i$ exists in l^q and

$$\lim_{a \rightarrow \infty} a^p P \left[\left\| \sum_{i \in A} \eta_i x_i \right\| > a \right] = c_p \sum_{i \in A} \|x_i\|^p \quad (6.4)$$

(here $\{\eta_i\}$ and $\{\theta_i\}$ are as above), but $(n^{-1/p}(Y_1 + \dots + Y_n))$ does not converge weakly. These examples are consistent with our previous results because l^q is not of stable type p for $p > q$.

Let $x_i = (0, \dots, 0, y_i, 0, \dots)$ where $y_i = i^{-2/q}$. We have $\sum_{i=1}^{\infty} \|x_i\|^p = \sum_{i=1}^{\infty} i^{-2p/q} < \infty$ since $p > q$. Also since $p > q$, $E|\theta|^q < \infty$ and we have $E[\|\sum_{i=1}^{\infty} \theta_i x_i\|_q] = E[\sum_{i=1}^{\infty} |\theta_i|^{q_i} i^{-2}] < \infty$. Therefore $\sum_{i=1}^{\infty} \theta_i x_i$ defines a stable measure on l^q .

Let $\{\rho_i\}$ be a sequence of independent random variables such that

$$\rho_i = \begin{cases} \pm y_i^{-1} & \text{each with prob. } (2i^{1+\varepsilon})^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad (6.5)$$

where $\varepsilon = (1 - q/p)/4$. It is clear that $X = \sum_{i=1}^{\infty} \rho_i x_i$ exists since $\sum_{i=1}^{\infty} P(\rho_i \neq 0) < \infty$. Note that $|\rho_i y_i| = 1$ or 0 so $|\rho_i y_i|^q = |\rho_i y_i|$. Using this fact and the Chebyshev inequality we have

$$P\left[\left\|\sum_{i=1}^{\infty} \rho_i x_i\right\| > a\right] \leq \exp(-a^q) E\left[\exp\left(\sum_{i=1}^{\infty} |\rho_i x_i|\right)\right] \\ \leq \exp(-a^q) C(\varepsilon)$$

where $C(\varepsilon)$ is a constant depending upon ε . It follows that

$$\lim_{a \rightarrow \infty} a^p P\left[\left\|\sum_{i \in A} \rho_i x_i\right\| > a\right] = 0. \quad (6.6)$$

Let $\{\theta_i\}$ be independent copies of θ such that $\{\theta_i\}$ and $\{\rho_i\}$ are also mutually independent. We define $\eta_i = \rho_i + \theta_i$. Clearly $Y = \sum_{i=1}^{\infty} \eta_i x_i$ exists and each η_i is in the domain of normal attraction of θ . It follows from (6.6) and Theorem 5.4 that (6.4) is satisfied. Also since $\{\rho_i\}$ and $\{\theta_i\}$ are independent, by considering characteristic functionals we see that $\sum_{i=1}^{\infty} \eta_i x_i$ is in the domain of normal attraction of $\sum_{i=1}^{\infty} \theta_i x_i$ if and only if $X = \sum_{i=1}^{\infty} \rho_i x_i$ is in the domain of normal attraction of δ_0 . We now show that

$$\mathcal{L}((X_1 + \cdots + X_n)/n^{1/p}) \not\rightarrow \delta_0, \quad n \rightarrow \infty, \quad (6.7)$$

where X_1, X_2, \dots are independent copies of X . To do this we use Theorem 3.1 [16].

Let $\{\xi_i\}$ be identically distributed, independent random variables with characteristic function $\exp - |t|^q$ such that $\{\xi_i\}$ and $\{\rho_i\}$ are mutually independent. (Note this is the same q as in l^q .) Let $(\Omega_1, \mathcal{F}_1, P_1)$ be the infinite product space generated by $\{\xi_i\}$ and let $\xi = (\xi_1, \xi_2, \dots)$. It follows from Theorem 3.1 [16] that a necessary condition for the law of S_n to converge weakly to δ_0 is that

$$E[e^{i\langle \xi, S_n \rangle}] \rightarrow 1, \quad (6.8)$$

a.s. with respect to P_1 measure on Ω_1 , where, in our case $S_n = n^{-1/p}(X_1 + \cdots + X_n)$. (Expectation is taken with respect to S_n .) We will conclude our example by contradicting (6.8).

By an easy computation we see that

$$E[e^{i\langle \xi, S_n \rangle}] = \prod_{i=1}^{\infty} \left[1 - i^{-(1+\varepsilon)} [1 - \cos \xi_i/n^{1/p}]\right]^n.$$

Thus to contradict (6.8) it is enough to show that

$$U_n \equiv \sum_{i=1}^{\infty} n i^{-(1+\varepsilon)} \left[1 - \cos \frac{\xi_i}{n^{1/p}}\right] \rightarrow 0 \quad (6.9)$$

P_1 -a.s. as $n \rightarrow \infty$. We have

$$U_n \geq \frac{n}{4} \sum_{i=1}^{\infty} i^{-(1+\varepsilon)} \frac{\xi_i^2}{n^{2/p}} I_{[\xi_i < n^{1/p}]}.$$

For $n \geq n_0$ for some integer n_0

$$\frac{\delta'(q)}{n^{q/p}} < P \left[\frac{1}{2} < \frac{|\xi_i|}{n^{1/p}} < 1 \right] < \frac{\delta''(q)}{n^{q/p}}$$

where $\delta'(q) = c_q(1 - \varepsilon')(2^q - 1)$, $\delta''(q) = c_q(1 + \varepsilon')(2^q - 1)$ for some $\varepsilon' > 0$ which can be made arbitrarily small depending on n_0 . Define $\{\varphi_i\}$ and Y_n as follows:

$$\varphi_i = \begin{cases} (|\xi_i|/n^{1/p})^2 & \text{if } \frac{1}{2} < |\xi_i|/n^{1/p} < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_n = \frac{n}{4} \sum_{i=n}^{\infty} i^{-(1+\varepsilon)} \varphi_i.$$

Note that U_n and Y_n are both defined on $(\Omega_1, \mathcal{F}_1, P_1)$ and that $U_n \geq Y_n$ pointwise on Ω_1 . Furthermore, $\frac{1}{4} < \varphi_i < 1$ on a set of measure equal to $\delta(q)/n^{q/p}$ where $\delta'(q) < \delta(q) < \delta''(q)$, otherwise $\varphi_i = 0$. We use the well-known inequality [12, p. 6]

$$P(Y_n > \lambda EY_n) \geq (1 - \lambda)^2 E^2 Y_n / EY_n^2 \quad (6.10)$$

where $0 < \lambda < 1$. Since $\delta'(q)/4n^{q/p} < E\varphi_i < \delta''(q)/n^{q/p}$ and $E\varphi_i^2 < \delta''(q)/n^{q/p}$ we get

$$\frac{\delta'(q, p)}{64} n^{(3/4)(1-q/p)} < EY_n < \frac{\delta''(q, p)}{16} n^{(3/4)(1-q/p)} \quad (6.11)$$

where $0 < \delta'(q, p) < \delta''(q, p) < \infty$. Also

$$EY_n^2 < (EY_n)^2 + E \left[\frac{n^2}{16} \sum_{i=n}^{\infty} i^{-2(1+\varepsilon)} \varphi_i^2 \right] \quad (6.12)$$

and

$$E \left[\frac{n^2}{16} \sum_{i=n}^{\infty} i^{-2(1+\varepsilon)} \varphi_i^2 \right] < \delta(p, q) n^{(1/2)(1-q/p)}$$

where $\delta(q, p) < \infty$ is a constant depending on both q and p . Since $p > q$, we can substitute (6.12) and (6.11) into (6.10) to see that, as $n \rightarrow \infty$, Y_n is unbounded on a set of positive measure. Since $U_n \geq Y_n$ pointwise we have obtained (6.9). This completes the demonstration.

Consider $X = \sum_{i=1}^{\infty} \rho_i x_i$. This satisfies condition (ii) of Theorem 5.1, yet X does not satisfy the weak law of large numbers, as we have just shown.

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